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by

Emilie Norton Martin

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ON THE IMPRIMITIVE SUBSTITUTION GROUPS OF DEGREE FIFTEEN AND THE PRIMITIVE SUBSTITUTION GROUPS OF DEGREE EIGHTEEN

A DISSERTATION

PRESENTED TO THE FACULTY OF BRYN MAWR COLLEGE FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

By EMILIE NORTON MARTIN

1901
The Lord Baltimore Press
THE FRIEDENWALD COMPANY
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On the Imprimitive Substitution Groups of Degree Fifteen and the Primitive Substitution Groups of Degree Eighteen.

By Emilie Norton Martin.

The following work is, with some slight modifications, the same as that of which an abstract was presented at the summer meeting of the American Mathematical Society in 1899. With regard to the imprimitive groups of degree fifteen, which form the subject matter of the first part of this paper, it should be stated that I have added two new groups to the list as originally presented, namely, the groups with five systems of imprimitivity simply isomorphic to the alternating and symmetric groups of degree 5, and that Dr. Kuhn reported at the February meeting of the Society, 1900, that he had carried the investigation further, adding 28 to the 70 groups that I succeeded in finding.

In the second part of this paper the determination of the primitive groups of degree 18 depends to a great extent upon the lists of transitive groups of lower degrees already determined. Any new discovery of groups of degree less than 18 would necessitate an examination of such groups to determine whether they can be combined with others in such a way as to generate a primitive group of degree 18. This list, therefore, cannot claim to be absolutely complete, since omissions are always possible.

Imprimitive Substitution Groups of Degree Fifteen.

Every imprimitive group contains a self-conjugate intransitive subgroup consisting of all the operations that interchange the elements of the systems of imprimitivity among themselves without interchanging the systems. Therefore, the problem of the determination of all imprimitive groups of degree 15 falls into two parts: 1st, the determination of all intransitive groups of degree 15

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capable of becoming the self-conjugate subgroups of such imprimitive groups; 2d, the determination of substitutions that will interchange the systems of imprimitivity and at the same time fulfill other conditions depending upon the particular group under discussion. The intransitive self-conjugate subgroup is called for shortness the *head*, the remaining substitutions of the imprimitive group are designated as the *tail*, a terminology that has been adopted by Dr. G. A. Miller in his papers on imprimitive groups.

The elements of an imprimitive group of degree 15 may fall into three systems of five elements each, or into five systems of three each. For the first of these cases, certain theorems given by Dr. G. A. Miller (Quar. Jour. Math., vol. XXVIII, 1896) are useful. With a slight modification in notation in order to adapt them to the notation of this paper, they are as follows, where G^1 represents a group in the elements with index 1, while G^2 and G^3 represent precisely the same group in the elements with indices 2 and 3.

THEOREM I.—*All the substitutions that can be used to construct tails are*

$$(a_1^1 a_2^1 \ldots a_n^1) \text{ all } (a_1^2 a_2^2 \ldots a_n^2) \text{ all } (a_1^3 a_2^3 \ldots a_n^3) \text{ all}$$
$$\{(a_1^1 a_1^2 a_1^3 . a_2^1 a_2^2 a_2^3 . \ldots . a_n^1 a_n^2 a_n^3), (a_1^1 a_1^2 . a_2^1 a_2^2 . \ldots . a_n^1 a_n^2)\}$$
$$- (a_1^1 a_2^1 \ldots a_n^1) \text{ all } (a_1^2 a_2^2 \ldots a_n^2) \text{ all } (a_1^3 a_2^3 \ldots a_n^3) \text{ all.}$$

THEOREM II.—*If* $G^1 = (a_1^1 a_2^1 \ldots a_n^1)$ all, *there are three imprimitive groups with the common head* $(G^1 G^2 G^3)$ pos, *and two with the common head* G^1 pos G^2 pos G^3 pos $+$ G^1 neg G^2 neg G^3 neg.

THEOREM III.—*If* $G^1 = (a_1^1 a_2^1 \ldots a_n^1)$ pos, *there are three imprimitive groups with the common head* $G^1 G^2 G^3$, *and three with the common head* $(G^1 G^2 G^3)_{1,1,1}$.

THEOREM IV.—*If the head is* $G^1 G^2 G^3$, *there is only one group which corresponds to* (abc) cyc.

The possible heads for these groups are got either by the direct multiplication of transitive groups of degree 5 in the three systems of elements, or by the establishment of isomorphic relations between such groups.

The transitive groups of degree 5 are five in number, and fall naturally into two categories, the first containing the symmetric group and its self-conjugate subgroup, the alternating group, the second containing the metacyclic group, together with its two self-conjugate transitive subgroups. These five groups are represented respectively by

$$(a_1 a_2 a_3 a_4 a_5) \text{ all, } (a_1 a_2 a_3 a_4 a_5) \text{ pos, } (a_1 a_2 a_3 a_4 a_5)_{20}, \ (a_1 a_2 a_3 a_4 a_5)_{10}, \ (a_1 a_2 a_3 a_4 a_5)_5 .$$

From the first two groups come the following heads:

I. $(a_1^1a_2^1a_3^1a_4^1a_5^1)$ all $(a_1^2a_2^2a_3^2a_4^2a_5^2)$ all $(a_1^3a_2^3a_3^3a_4^3a_5^3)$ all $= H_{1728000}$.

II. $\{(a_1^1a_2^1a_3^1a_4^1a_5^1)$ all $(a_1^2a_2^2a_3^2a_4^2a_5^2)$ all $(a_1^3a_2^3a_3^3a_4^3a_5^3)$ all$\}$ pos $= H_{864000}$.

III. $(a_1^1a_2^1a_3^1a_4^1a_5^1)$ pos $(a_1^2a_2^2a_3^2a_4^2a_5^2)$ pos $(a_1^3a_2^3a_3^3a_4^3a_5^3)$ pos
$+ (a_1^1a_2^1a_3^1a_4^1a_5^1)$ neg $(a_1^2a_2^2a_3^2a_4^2a_5^2)$ neg $(a_1^3a_2^3a_3^3a_4^3a_5^3)$ neg $= H_{432000}$.

IV. $(a_1^1a_2^1a_3^1a_4^1a_5^1)$ pos $(a_1^2a_2^2a_3^2a_4^2a_5^2)$ pos $(a_1^3a_2^3a_3^3a_4^3a_5^3)$ pos $= H_{216000}$.

V. $(a_1^1a_2^1a_3^1a_4^1a_5^1 \,.\, a_1^2a_2^2a_3^2a_4^2a_5^2 \,.\, a_1^3a_2^3a_3^3a_4^3a_5^3)$ all $= H_{120}$.

VI. $(a_1^1a_2^1a_3^1a_4^1a_5^1 \,.\, a_1^2a_2^2a_3^2a_4^2a_5^2 \,.\, a_1^3a_2^3a_3^3a_4^3a_5^3)$ pos $= H_{60}$.

From the three remaining groups come the heads:

VII. $(a_1^1a_2^1a_3^1a_4^1a_5^1)_{20}\,(a_1^2a_2^2a_3^2a_4^2a_5^2)_{20}\,(a_1^3a_2^3a_3^3a_4^3a_5^3)_{20} = H_{8000}$.

VIII. $\{(a_1^1a_2^1a_3^1a_4^1a_5^1)_{20}\,(a_1^2a_2^2a_3^2a_4^2a_5^2)_{20}\,(a_1^3a_2^3a_3^3a_4^3a_5^3)_{20}\}$ pos $= H_{4000}$.

IX. $\{(a_1^1a_2^1a_3^1a_4^1a_5^1)_{20}\,(a_1^2a_2^2a_3^2a_4^2a_5^2)_{20}\,(a_1^3a_2^3a_3^3a_4^3a_5^3)_{20}\}_{10,\ 10,\ 10} = H_{2000}$.

X. $(a_1^1a_2^1a_3^1a_4^1a_5^1)_{10}\,(a_1^2a_2^2a_3^2a_4^2a_5^2)_{10}\,(a_1^3a_2^3a_3^3a_4^3a_5^3)_{10} = H_{1000}$.

XI. $\{(a_1^1a_2^1a_3^1a_4^1a_5^1)_{20}\,(a_1^2a_2^2a_3^2a_4^2a_5^2)_{20}\,(a_1^3a_2^3a_3^3a_4^3a_5^3)_{20}\}_{5,\ 5,\ 5} = H_{500}$.

XII. $\{(a_1^1a_2^1a_3^1a_4^1a_5^1)_{10}\,(a_1^2a_2^2a_3^2a_4^2a_5^2)_{10}\,(a_1^3a_2^3a_3^3a_4^3a_5^3)_{10}\}_{5,\ 5,\ 5} = H_{250}$.

XIII. $(a_1^1a_2^1a_3^1a_4^1a_5^1)$ cyc $(a_1^2a_2^2a_3^2a_4^2a_5^2)$ cyc $(a_1^3a_2^3a_3^3a_4^3a_5^3)$ cyc $= H_{125}$.

XIV. $(a_1^1a_2^1a_3^1a_4^1a_5^1 \,.\, a_1^2a_2^2a_3^2a_4^2a_5^2 \,.\, a_1^3a_2^3a_3^3a_4^3a_5^3)_{20} = H_{20}$.

XV. $(a_1^1a_2^1a_3^1a_4^1a_5^1 \,.\, a_1^2a_2^2a_3^2a_4^2a_5^2 \,.\, a_1^3a_2^3a_3^3a_4^3a_5^3)_{10} = H_{10}$.

XVI. $(a_1^1a_2^1a_3^1a_4^1a_5^1 \,.\, a_1^2a_2^2a_3^2a_4^2a_5^2 \,.\, a_1^3a_2^3a_3^3a_4^3a_5^3)$ cyc. $= H_5$.

The groups corresponding to these heads may be isomorphic either to $(a^1a^2a^3)$ cyc or to $(a^1a^2a^3)$ all. To generate a group isomorphic to $(a^1a^2a^3)$ cyc a substitution with the following properties must be added to the head: it must have its cube in the head, it must interchange all three systems, and it must transform the head into itself. Calling the group so found G, the groups isomorphic to $(a^1a^2a^3)$ all may be found by combining with G any substitution that has its square in the head, that interchanges two of the systems leaving the third unaffected, and that transforms H into itself, and G into itself.

As all the heads given above are symmetric in the three sets of elements, each head furnishes two groups by means of the symmetrically formed substitutions

$$s = a_1^1a_1^2a_1^3 \,.\, a_2^1a_2^2a_2^3 \,.\, a_3^1a_3^2a_3^3 \,.\, a_4^1a_4^2a_4^3 \,.\, a_5^1a_5^2a_5^3, \quad t = a_1^1a_1^2 \,.\, a_2^1a_2^2 \,.\, a_3^1a_3^2 \,.\, a_4^1a_4^2 \,.\, a_5^1a_5^2.$$

The letters s and t are used throughout this section of the paper to denote these particular substitutions, other substitutions fulfilling the same conditions being denoted by the same letters with suffixes.

According to Theorem I, any s_a or t_a must be the product of some substitution, σ_a, of the most general head, $H_{1728000}$, by s or t. Therefore σ_a must be a substitution of a subgroup of $H_{1728000}$ that contains the special H under consideration as a self-conjugate subgroup.

We may now proceed to the determination of the groups to be derived from the various heads taken in order.

I. $H_{1728000}$ gives us, according to Theorem I, only the two groups,

$$\{H_{1728000}, s,\} \text{ of order } 5184000_1,$$
$$\text{and } \{H_{1728000}, s, t\} \text{ of order } 10368000.$$

II. H_{864000} gives us, in accordance with Theorem II, three distinct groups. Of these, two are the groups,

$$\{H_{864000}, s\} \text{ of order } 2592000_1,$$
$$\{H_{864000}, s, t\} \text{ of order } 5184000_2,$$

A σ that transforms the head into itself without belonging in the head is $\sigma = a_1^1 a_2^1$. This cannot be combined with s, as $(\sigma s)^3$ is an odd substitution; it may, however, be combined with t. The remaining group is therefore

$$\{H_{864000}, s, a_1^1 a_2^1 . t\} \text{ of order } 5184000_3.$$

Of these two groups of order 5184000, the first contains both odd and even substitutions, the second only even.

III. H_{432000} gives, by Theorem II, the two groups

$$\{H_{432000}, s\} \text{ of order } 1296000_1,$$
$$\{H_{432000}, s, t\} \text{ of order } 2592000_2.$$

IV. H_{216000} gives us, by Theorem III, three distinct groups. $\sigma = a_1^1 a_2^1$ transforms the head into itself, but when combined with s it gives an odd substitution whose cube cannot be found in the head. The substitution σt furnishes us however, with a new t_a. The three groups are, therefore,

$$\{H_{216000}, s\} \text{ of order } 648000,$$
$$\{H_{216000}, s, t\} \text{ of order } 1296000_2,$$
$$\{H_{216000}, s, a_1^1 a_2^1 . t\} \text{ of order } 1296000_3.$$

The two groups of order 1296000 are distinct, since the one contains both odd and even substitutions, the other only even.

V. H_{120} is not contained self-conjugately in any larger subgroup of $H_{1728000}$, therefore only the two following groups can be formed from it:

$$\{H_{120}, s\} \text{ of order } 360_1,$$
$$\{H_{120}, s, t\} \text{ of order } 720.$$

VI. H_{60} gives, in accordance with Theorem III, three groups:

$$\{H_{60}, s\} \text{ of order } 180,$$
$$\{H_{60}, s, t\} \text{ of order } 360_2,$$
$$\{H_{60}, s, a_1^1 a_2^1 . a_1^2 a_2^2 . a_1^3 a_2^3 . t\} \text{ of order } 360_3.$$

The last of these groups consists entirely of even substitutions.

The remaining heads are all composed of substitutions of the type

$$v_{a^1}^{\alpha} u_{a^1}^{\alpha'} v_{a^2}^{\beta} u_{a^2}^{\beta'} v_{a^3}^{\gamma} u_{a^3}^{\gamma'}, \tag{1}$$

where $v_{a^1} = a_2^1 a_3^1 a_5^1 a_4^1$, $u_{a^1} = a_1^1 a_2^1 a_3^1 a_4^1 a_5^1$, while v_{a^2}, u_{a^2}, v_{a^3}, u_{a^3} denote the same substitutions written in elements with the indices 2 and 3 respectively. The substitutions v_{a^1}, u_{a^1} generate the metacyclic group in the five elements with index 1, these substitutions being subject to the conditions

$$v_{a^1}^4 = 1, \quad u_{a^1}^5 = 1, \quad u_{a^1}^x v_{a^1}^y = v_{a^1}^y u_{a^1}^{y^y x}.$$

The most general s_α is given by

$$s_\alpha = v_{a^1}^{a} u_{a^1}^{a_1} v_{a^2}^{b} u_{a^2}^{b_1} v_{a^3}^{c} u_{a^3}^{c_1} s. \tag{2}$$

From this we find

$$s_\alpha^3 = v_{a^1}^{a+b+c} u_{a^1}^{\lambda} v_{a^2}^{a+b+c} u_{a^2}^{\mu} v_{a^3}^{a+b+c} u_{a^3}^{\nu}, \tag{3}$$

where

$$\left.\begin{aligned} \lambda &= 2^c(2^b a_1 + b_1) + c_1, \\ \mu &= 2^a(2^c b_1 + c_1) + a_1, \\ \nu &= 2^b(2^a c_1 + a_1) + b_1. \end{aligned}\right\} \tag{4}$$

Transformation of the general substitution (1) by s_α gives us

$$s_\alpha^{-1} v_{a^1}^{\alpha} u_{a^1}^{\alpha'} v_{a^2}^{\beta} u_{a^2}^{\beta'} v_{a^3}^{\gamma} u_{a^3}^{\gamma'} s_\alpha = v_{a^1}^{\gamma_1} u_{a^1}^{\nu} v_{a^2}^{\alpha} u_{a^2}^{\lambda} v_{a^3}^{\beta} u_{a^3}^{\mu}, \tag{5}$$

where

$$\left.\begin{aligned} \lambda &= a_1 + 2^a\alpha' - 2^a a_1, \\ \mu &= b_1 + 2^b\beta' - 2^\beta b_1, \\ \nu &= c_1 + 2^c\gamma' - 2^\gamma c_1, \end{aligned}\right\} \tag{6}$$

The general substitution of the group $G = \{H, s_\alpha\}$ is

$$T = s_\alpha^r v_{a^1}^{\alpha} u_{a^1}^{\alpha'} v_{a^2}^{\beta} u_{a^2}^{\beta'} v_{a^3}^{\gamma} u_{a^3}^{\gamma'}. \tag{7}$$

The most general t_β is given by

$$t_\beta = v_{a^1}^{a_2} u_{a^1}^{a_3} v_{a^2}^{b_2} u_{a^2}^{b_3} v_{a^3}^{c_2} u_{a^3}^{c_3} t. \tag{8}$$

Upon squaring this substitution, we get

$$t_\beta^2 = v_{a^1}^{a_2 + b_2} u_{a^1}^{\lambda} v_{a^2}^{a_2 + b_2} u_{a^2}^{\mu} v_{a^3}^{2c_2} u_{a^3}^{\nu}, \tag{9}$$

where

$$\left.\begin{aligned} \lambda &= 2^{b_2} a_3 + b_3, \\ \mu &= 2^{a_2} b_3 + a_3, \\ \nu &= 2^{c_2} c_3 + c_3. \end{aligned}\right\} \tag{10}$$

On transforming the general substitution T by the general t_β, we have, after a straight-forward calculation, the following expression for the case $x = 1$:

$$t_\beta^{-1} T_{x=1} t_\beta = s^2 v_{a^1}^{-a_2 + b_2 + \beta + a} u_{a^1}^{\lambda} v_{a^2}^{a_2 - c_2 + a + c} u_{a^2}^{\mu} v_{a^3}^{-b_2 + c_2 + \gamma + b} u_{a_3}^{\nu}, \tag{11}$$

where

$$\left.\begin{aligned} \lambda &= -2^{b_2 - a_2 + \beta + a} a_3 + 2^{\beta + b_2} a_1 + 2^{b_2} \beta' + b_3, \\ \mu &= -2^{a_2 - c_2 + a + c} c_3 + 2^{a + a_2} c_1 + 2^{a_2} \alpha' + a_3, \\ \nu &= -2^{c_2 - b_2 + \gamma + b} b_3 + 2^{\gamma + c_2} b_1 + 2^{c_2} \gamma' + c_3. \end{aligned}\right\} \tag{12}$$

We may now return to the consideration of special groups.

VII. H_{8000} gives only the two groups formed with s and t, as any σ that might be used is already contained in this head. The groups are, therefore,

$$\{H_{8000}, s\} \text{ of order } 24000_1,$$
$$\{H_{8000}, s, t\} \text{ of order } 48000.$$

VIII. H_{4000} has the general substitution (1) subject to the condition $\alpha + \beta + \gamma \equiv 0 \pmod 2$. From (3), it is evident that s_a is subject to the condition $a + b + c \equiv 0 \pmod 2$. Therefore, s_a is already in the group generated by H_{4000} and by s, and there is only one group isomorphic to $(a^1 a^2 a^3)$ cyc. We find by (9) that every t_β has its square in the head, and by (11), that every t_β transforms the head into itself, therefore, we may take as a new t_β the simplest substitution for which $a_2 + b_2 + c_2 \equiv 1 \pmod 2$, viz.:

$$v_a t = a_1^1 a_1^2 . a_2^1 a_3^2 a_3^1 a_5^2 a_5^1 a_4^2 a_4^1 a_2^2.$$

The three groups with this head are, therefore,

$$\{H_{4000}, s\} \text{ of order } 12000_1,$$
$$\{H_{4000}, s, t\} \text{ of order } 24000_2,$$
$$\{H_{4000}, s, v_a t\} \text{ of order } 24000_3.$$

Of these groups the first and third consist of even substitutions, the second of even and odd.

IX. H_{2000} has the general substitution subject to the condition $\alpha \equiv \beta \equiv \gamma \pmod 2$. From (3) and (5), it is plain that every s_α can be used to generate a group of the kind required. The only possible form for the cofactor of s, if it is not to give the group generated by s and the head, is $v_{a^1}^a v_{a^2}^b v_{a^3}^c$, where a, b, c do not fulfill the condition $a \equiv b \equiv c \pmod 2$. The simplest form for such a cofactor, and a form to which all others reduce, is found by making two of the exponents vanish and the third become equal to 1, e. g., $s_1 = v_{a^1} s = a_1^1 a_1^2 a_1^3 \,.\, a_2^1 a_3^2 a_3^3 a_3^1 a_5^2 a_5^3 a_5^1 a_4^2 a_4^3 a_4^1 a_2^2 a_2^3$. Now, $s_1^4 = s_1 \,.\, v_{a^1} v_{a^2} v_{a^3}$ and $s_1^8 = s_1^2 \,.\, v_{a^1}^2 v_{a^2}^2 v_{a^3}^2$; we may, therefore, take s_1^4 as the s. in the place of s_1 and still have the same group. But $s_1^4 = (v_{a^2}^2 v_{a^3}^3)^{-1} s \, (v_{a^2}^2 v_{a^3}^3)_a$ therefore the group we have now found is merely the transformed of the group generated by s with respect to the substitution $v_{a^2}^2 v_{a^3}^3$. Consequently, there is but one group corresponding to the cyclic group of degree three.

If, in addition to the group given by t, we have a group given by t_β, then according to the relations derived from (11), $a_2 \equiv b_2 \equiv c_2 \pmod 2$, i. e., the possible values of t_β are already present in the group generated with the help of t. The two imprimitive groups with this head are, therefore, the groups

$$\{H_{2000}, s\} \text{ of order } 6000.$$
$$\{H_{2000}, s, t\} \text{ of order } 12000_2.$$

In this, and all following work, the terms u in the cofactors of s and t are taken as unity, unless the contrary is expressly stated.

X. H_{1000} has its general substitution subject to the condition $\alpha \equiv \beta \equiv \gamma \equiv 0 \pmod 2$. By Theorem IV, this head gives only one group isomorphic to (abc) cyc. If, in addition to the substitution t, there is a substitution t_β, the relations satisfied by the exponents of the v's in (11) reduces to $a_2 \equiv b_2 \equiv c_2 \pmod 2$. We have, therefore, two distinct groups according as a_2 is even or odd. The three groups with this head are

$$\{H_{1000}, s\} \text{ of order } 3000_1,$$
$$\{H_{1000}, s, t\} \text{ of order } 6000_2,$$
$$\{H_{1000}, s, v_{a^1} v_{a^2} v_{a^3} t\} \text{ of order } 6000_3.$$

XI. H_{500} subjects the general substitution to the conditions $\alpha = \beta = \gamma$, where $\alpha = 0, 1, 2, 3$. Since every substitution s_α satisfies the necessary conditions, the following independent types of s_α must be examined: $v_{a^1} s$, $v_{a^1}^2 s$, $v_{a^1}^3 s$, $v_{a^1} v_{a^2}^2 s$. The

fourth power of these substitutions is in every case the transformed of s with respect to some combination of the v's; therefore, they give nothing new. The possible forms for t_β are derived from the equation easily deducible from (11); $-a_2+b_2\equiv a_2-c_2\equiv -b_2+c_2 \pmod 4$, which, taken in conjunction with the limited range of values of a_2, b_2, c_2, gives $a_2=b_2=c_2$. That is, every possible t_β is already included in the group generated by t. This head gives accordingly only the two groups,

$$\{H_{500}, s\} \text{ of order } 1500_1,$$
$$\{H_{500}, s, t\} \text{ of order } 3000_2.$$

XII. H_{250} subjects the general substitution (1) to the conditions $\alpha=\beta=\gamma\equiv 0 \pmod 2$. To determine an s_α, we have from (3) the condition $a+b+c\equiv 0 \pmod 2$. An examination of the four apparently distinct types of s_α, $v_1^2 s$, $v_1 v_2 s$, $v_1^3 v_2 s$, $v_1^3 v_2^3 s$, shows that just as in the last set of groups, these each give a group that can be derived from the group generated by s by means of an easy transformation.

The possible forms t_β must fulfill the conditions, deducible from (11), $-a_2+b_2\equiv -b_2+c_2\equiv -c_2+a_2\equiv 0 \pmod 2$ and also $-a_2+b_2\equiv a_2-c_2 \pmod 4$. These reduce to the simple condition $a_2=b_2=c_2$, which furnishes the substitution $t_\beta = v_{a^1} v_{a^2} v_{a^3} t$. This head gives therefore the three groups,

$$\{H_{250}, s\} \text{ of order } 750_1,$$
$$\{H_{250}, s, t\} \text{ of order } 1500_2,$$
$$\{H_{250}, s, v_{a^1} v_{a^2} v_{a^3} t\} \text{ of order } 1500_3.$$

The second group alone contains odd substitutions.

XIII. H_{125} gives in accordance with Theorem IV only one group in which the systems are interchanged cyclically. The general substitution of this head is subject to the condition $\alpha=\beta=\gamma=0$. Applying this condition to (9) and (11) we find $a_2=b_2=c_2$, while a_2 lies under the further restriction of being even. Therefore we have in addition to t the substitution,

$$t_\beta = v_{a^1}^2 v_{a^2}^2 v_{a^3}^2 t = a_1^1 a_1^2 . a_2^1 a_5^2 . a_3^1 a_4^2 . a_4^1 a_3^2 . a_5^1 a_2^2 . a_2^3 a_5^3 . a_3^3 a_4^3 .$$

The three groups given by this head are,

$$\{H_{125}, s\} \text{ of order } 375,$$
$$\{H_{125}, s, t\} \text{ of order } 750_2,$$
$$\{H_{125}, s, v_{a^1}^2 v_{a^2}^2 v_{a^3}^2 t\} \text{ of order } 750_3.$$

XIV. H_{20} imposes upon the exponents of the general term the conditions

$\alpha = \beta = \gamma$, $\alpha' = \beta' = \gamma'$. Making use of this in (5) and (6) we find $2^a \alpha' \equiv 2^b \alpha' \equiv 2^c \alpha'$ (mod 5), which gives at once $a = b = c$. Using this latter equality in the equations that are deduced from (3) and (4) we find $a_1 = b_1 = c_1$ with the single exception of the case $a = 0$, where the equations become indeterminate, being satisfied by every value of a_1, b_1, c_1. An examination of all of the apparently independent sets of value for a_1, b_1, c_1 shows that in every case the group is transformable into that generated by s alone. In order to determine all substitutions t_β we use the equation, derived from (11), $-a_2 + b_2 \equiv a_2 - c_2 \equiv c_2 - b_2$ (mod. 4), from which follows at once $a_2 = b_2 = c_2$. From (12), by making use of the special case $\alpha = \beta = \gamma = 0$, can be derived the relations $-a_3 + b_3 \equiv -c_3 + a_3 \equiv -b + c_3$ (mod. 5); i, e, $a_3 = b_3 = c_3$. The only groups with this head are therefore the two groups,

$$\{H_{20}, s\} \text{ of order } 60_1,$$
$$\{H_{20}, s, t\} \text{ of order } 120.$$

XV. H_{10} has the general term (1) subject to the conditions $\alpha = \beta = \gamma \equiv 0$ (mod 2), $\alpha' = \beta' = \gamma'$. By precisely the same line of argument as that laid down in the preceding case we arrive at the conclusion $a = b = c$, $a_1 = b_1 = c_1$, $a_2 = b_2 = c_2$, $a_3 = b_3 = c_3$. In this work, too, the indeterminate values of a_1, b_1, c_1 require a careful examination that leads to no new group. From this head come, therefore, the three groups,

$$\{H_{10}, s\} \text{ of order } 30_1,$$
$$\{H_{10}, s, t\} \text{ of order } 60_2,$$
$$\{H_{10}, s, v_{a^1} v_{a^2} v_{a^3} t\} \text{ of order } 60_3.$$

Of these three groups the second alone involves odd substitutions.

XVI. H_5 imposes upon the general term the conditions $\alpha = \beta = \gamma = 0$, $\alpha' = \beta' = \gamma'$, By arguments similar to those used in the last two cases, with the further addition of the condition imposed by (3), $a + b + c \equiv 0$ (mod 4), we find $a = b = c = 0$, $a_1 = b_1 = c_1$. In the determination of t_β we see at once from (9) that c_2 must be even, while from (11) we find $a_2 = b_2 = c_2$, and from (12) $a_3 = b_3 = c_3$.

The groups given by this head are as follows:

$$\{H_5, s\} \text{ of order } 15,$$
$$\{H_5, s, t\} \text{ of order } 30_2,$$
$$\{H_5, s, v_{a^1}^2 v_{a^2}^2 v_{a^3}^2 t\} \text{ of order } 30_3.$$

Passing now to the case of five systems of three elements each, there are seven heads considered in this paper, six involving all the systems symmetrically, the remaining head being unity.

I. $(a_1^1 a_2^1 a_3^1)$ all $(a_1^2 a_2^2 a_3^2)$ all $(a_1^3 a_2^3 a_3^3)$ all $(a_1^4 a_2^4 a_3^4)$ all $(a_1^5 a_2^5 a_3^5)$ all $= H_{7776}$,

II. $\{H_{7776}\}$ pos $= H_{3888}$,

III. $\{H_{7776}\}_{3,3,3,3,3} = H_{486}$,

IV. $(a_1^1 a_2^1 a_3^1)$ pos $(a_1^2 a_2^2 a_3^2)$ pos $(a_1^3 a_2^3 a_3^3)$ pos $(a_1^4 a_2^4 a_3^4)$ pos $(a_1^5 a_2^5 a_3^5)$ pos $= H_{243}$,

V. $(a_1^1 a_2^1 a_3^1 \,.\, a_1^2 a_2^2 a_3^2 \,.\, a_1^3 a_2^3 a_3^3 \,.\, a_1^4 a_2^4 a_3^4 \,.\, a_1^5 a_2^5 a_3^5)$ all $= H_6$,

VI. $(a_1^1 a_2^1 a_3^1 \,.\, a_1^2 a_2^2 a_3^2 \,.\, a_1^3 a_2^3 a_3^3 \,.\, a_1^4 a_2^4 a_3^4 \,.\, a_1^5 a_2^5 a_3^5)$ cyc $= H_3$,

VII. Unity.

Denoting the system with index r by A_r, it is evident that these systems must be interchanged according to the five groups $(A_1A_2A_3A_4A_5)$ cyc, $(A_1A_2A_3A_4A_5)_{10}$, $(A_1A_2A_3A_4A_5)_{20}$, $(A_1A_2A_3A_4A_5)$ pos, $(A_1A_2A_3A_4A_5)$ all.

The order of procedure in each case is as follows:

1. If the group is to correspond to $(A_1A_2A_3A_4A_5)$ cyc, a substitution s must be found that will interchange the systems cyclically, transform the head into itself, and have its fifth power in the head. The imprimitive group so generated may be called G.

2. If the group is to correspond to $(A_1A_2A_3A_4A_5)_{10}$, it must contain G_1 as a self-conjugate subgroup. In addition, therefore, to the s of case 1, a substitution t must be found that will interchange four of the systems in two pairs, as $A_2A_5 \,.\, A_3A_4$, while leaving the remaining system unaltered, and that will, at the same time, transform the head into itself and G into itself. This substitution t must also have its square in the head. This imprimitive group shall be called G_2.

3. If the group is to correspond to $(A_1A_2A_3A_4A_5)_{20}$, it must contain both G_1 and G_2 as self-conjugate subgroups. In addition, therefore, to the s of case 1, a substitution u must be found interchanging four of the systems cyclically, according to $A_2A_3A_5A_4$ for instance, transforming G_1 and G_2 into themselves and having its fourth power in the head.

4. If the group is to correspond to $(A_1A_2A_3A_4A_5)$ pos, two substitutions, v and v', must be found corresponding to $A_1A_2A_3$ and $A_1A_4A_5$. These sub-

stitutions must, therefore, each interchange three systems, leaving two unaltered, they must have their cubes in the head, and must transform the head into itself. This group may be called G'.

5. If the group is to correspond to $(A_1A_2A_3A_4A_5)$ all, two substitutions, w and w', must be found corresponding to $A_1A_2A_3A_4$ and A_1A_5. G' is to be contained in this new group as a self-conjugate subgroup, therefore w and w' must transform the head into itself and G' into itself. The fourth power of w and the square of w' must both be contained in the head.

I. H_{7776} is the largest possible intransitive group with the given systems of intransitivity, and, consequently, only one group with this head corresponds to each of the transitive groups of degree 5. For each of these groups a substitution or pair of substitutions can be found fulfilling all required conditions and involving the elements of the systems symmetrically. A second set could be found only by multiplying this first step by some substitution belonging to the largest group that contains the head self-conjugately without interchanging any of the systems. But this group is the head itself. The required groups are, therefore, the following:

$$\{H_{7776},\ s\} \text{ of order } 38880,$$
$$\{H_{7776},\ s,\ t\} \text{ of order } 77760,$$
$$\{H_{7776},\ s,\ u\} \text{ of order } 155520,$$
$$\{H_{7776},\ v,\ v'\} \text{ of order } 466560_1,$$
$$\{H_{7776},\ w,\ w'\} \text{ of order } 933120,$$

where

$$s = a_1^1a_1^2a_1^3a_1^4a_1^5 \,.\, a_2^1a_2^2a_2^3a_2^4a_2^5 \,.\, a_3^1a_3^2a_3^3a_3^4a_3^5,$$
$$t = a_1^2a_1^5 \,.\, a_1^3a_1^4 \,.\, a_2^2a_2^5 \,.\, a_2^3a_2^4 \,.\, a_3^2a_3^5 \,.\, a_3^3a_3^4,$$
$$u = a_1^2a_1^3a_1^5a_1^4 \,.\, a_2^2a_2^3a_2^5a_2^4 \,.\, a_3^2a_3^3a_3^5a_3^4,$$
$$v = a_1^1a_1^2a_1^3 \,.\, a_2^1a_2^2a_2^3 \,.\, a_3^1a_3^2a_3^3,$$
$$v' = a_1^1a_1^4a_1^5 \,.\, a_2^1a_2^4a_2^5 \,.\, a_3^1a_3^4a_3^5,$$
$$w = a_1^1a_1^2a_1^3a_1^4 \,.\, a_2^1a_2^2a_2^3a_2^4 \,.\, a_3^1a_3^2a_3^3a_3^4,$$
$$w' = a_1^1a_1^5 \,.\, a_2^1a_2^5 \,.\, a_3^1a_3^5.$$

These letters shall be kept throughout this section of the paper to denote these symmetrically formed substitutions, other substitutions with corresponding properties being denoted by the same letters with suffixes.

II. H_{3888} gives only one group isomorphic to $(A_1A_2A_3A_4A_5)$ cyc, viz., the group generated by s. Any new s_a must have as cofactor an odd substitution belonging to H_{7776}, but the fifth power of such a substitution is not contained in the head. There are, however, two groups isomorphic to $(A_1A_2A_3A_4A_5)_{10}$, since both t and $t_a = a_1^1a_2^1 . t$ fulfill the necessary conditions. The former generates a group G_{38880_2} containing only even substitutions, the latter generates a group G_{38880_3} containing both odd and even substitutions. There are likewise two groups isomorphic to $(A_1A_2A_3A_4A_5)_{20}$, one generated by u, the other by $a_1^1a_2^1 . u$. The first of these groups contains odd substitutions, the second only even. G_{38880_2} is contained self-conjugately in both.

Only one group G' can be found for this head, as no new v_a or v_a' fulfills the necessary conditions. Such a substitution would necessarily be of the form σv or $\sigma v'$, where σ would belong to the group H_{7776}. If σ were even, the group so generated would be a repetition of the group generated by v and v'. If σ were odd, the cubes of σv, $\sigma v'$ would not be contained in the head.

Two groups can be found isomorphic to $(A_1A_2A_3A_4A_5)$ all, the substitutions w and w' generating one group, the substitutions $a_1^1a_2^1 . w$, $a_1^1a_2^1 . w'$ generating the other. This latter group contains only even substitutions.

From this head we have, therefore, derived eight groups:

$\{H_{3888}, s\}$ of order 19440,
$\{H_{3888}, s . t\}$ of order 38880_2,
$\{H_{3888}, s, a_1^1a_2^1 . t\}$ of order 38880_3,
$\{H_{3888}, s, u\}$ of order 76660_2,
$\{H_{3888}, s, a_1^1a_2^1 . u\}$ of order 76660_3,
$\{H_{3888}, v, v'\}$ of order 233280,
$\{H_{3888}, w, w'\}$ of order 466560_2,
$\{H_{3888}, a_1^1a_2^1 . w, a_1^1a_2^1 . w'\}$ of order 466560_3.

III. H_{456} furnishes us with only one group isomorphic to $(A_1A_2A_3A_4A_5)$ cyc, for an examination of the groups given by all possible types of substitutions s_a shows that each of these groups is merely the group generated by the help of s and transformed with respect to some easily discovered substitution. Moreover, there is but one group isomorphic to $(A_1A_2A_3A_4A_5)_{10}$, viz., that generated with the help of t. Any cofactor of t must be of one of the types $a_1^1a_2^1$, $a_1^2a_2^2 . a_1^5a_2^5$, $a_1^1a_2^1 . a_1^2a_2^2 . a_1^5a_2^5$, $a_1^2a_2^2 . a_1^3a_2^3 . a_1^4a_2^4 . a_1^5a_2^5$, but any t_a got by means of these, transforms

s into s^4 (a substitution not in the head). Precisely the same reasoning shows that there is only the one group isomorphic to $(A_1A_2A_3A_4A_5)_{20}$.

In addition to the group isomorphic to $(A_1A_2A_3A_4A_5)$ pos generated by means of the substitutions v and v', we must examine groups generated with the help of v_α and v'_α, substitutions which contain as cofactors of v, v' respectively the products of transposition, one transposition from each system. A number of these may be rejected at once, but we are left with the possible forms:

$$v_1 = a_1^1a_2^1 \,.\, a_1^2a_2^2 \,.\, v = a_1^1a_2^2a_1^3 \,.\, a_2^1a_1^2a_2^3 \,.\, a_3^1a_3^2a_3^3,$$
$$v_2 = a_1^1a_2^1 \,.\, a_1^4a_2^4 \,.\, a_1^5a_2^5 \,.\, v = a_1^1a_2^2a_2^3 \,.\, a_2^1a_1^2a_1^3 \,.\, a_1^4a_2^4 \,.\, a_1^5a_2^5,$$
$$v'_1 = a_1^1a_2^1 \,.\, a_1^4a_2^4 \,.\, v' = a_1^1a_2^4a_1^5 \,.\, a_2^1a_1^4a_2^5 \,.\, a_3^1a_3^4a_3^5,$$
$$v'_2 = a_1^1a_2^1 \,.\, a_1^2a_2^2 \,.\, a_1^3a_2^3 \,.\, v' = a_1^1a_2^4a_2^5 \,.\, a_2^1a_1^4a_1^5 \,.\, a_1^2a_2^2 \,.\, a_1^3a_2^3.$$

But, since v_1, v_2^4 are transformable into v, and v'_1, $v_2'^4$ are transformable into v', it is impossible to generate any group by means of any combination of these four substitutions excepting a group that can be transformed into the one generated by means of v and v'.

A similar examination of all groups isomorphic to $(A_1A_2A_3A_4A_5)$ all, shows that, in addition to the group generated with the help of w and w', there is one other group generated by means of $w_\alpha = a_1^5a_2^5 \,.\, w$ and w'.

From this head are therefore formed the six following groups:

$\{H_{486}, s\}$ of order 2430_1,
$\{H_{486}, s, t\}$ of order 4860_1,
$\{H_{486}, s, u\}$ of order 9720,
$\{H_{486}, v, v'\}$ of order 29160_1,
$\{H_{486}, w, w'\}$ of order 58320_1,
$\{H_{436}, a_1^5a_2^5 \,.\, w, w'\}$ of order 58320_2.

IV. H_{243} gives one group isomorphic to $(A_1A_2A_3A_4A_5)$ cyc, by means of s. The only other permissible forms of s_α are of the type

$$s_1 = a_1^1a_2^1 \,.\, a_1^2a_2^2 \,.\, s, \quad s_2 = a_1^1a_2^1 \,.\, a_1^2a_2^2 \,.\, a_1^3a_2^3 \,.\, a_1^4a_2^4 \,.\, s.$$

But s_1 and s_2 are each the transformed of s with respect to some substitution that transforms the head into itself; therefore, there is only the one group of this type. On the other hand, there are two groups isomorphic to $(A_1A_2A_3A_4A_5)_{10}$, since both t and $t_1 = a_1^1a_2^1 \,.\, a_1^2a_2^2 \,.\, a_1^3a_2^3 \,.\, a_1^4a_2^4 \,.\, a_1^5a_2^5 \,.\, t$ fulfill all necessary conditions and generate, one a group of even substitutions, the other a group containing odd

substitutions. There are also two groups isomorphic to $(A_1 A_2 A_3 A_4 A_5)_{20}$, one containing both odd and even substitutions, the other only even. These are generated respectively by means of u and of $u_1 = a_1^1 a_2^1 . a_1^2 a_2^2 . a_1^3 a_2^3 . a_1^4 a_2^4 . a_1^5 a_2^5 . u$, and each contains as a self-conjugate subgroup the group isomorphic to $(A_1 A_2 A_3 A_4 A_5)_{10}$ that consists entirely of even substitutions.

Only one group can be found isomorphic to $(A_1 A_2 A_3 A_4 A_5)$ pos, and this is the one formed by the help of v and v'. An examination of the various substitutions v_a and v_a' corresponding to various types of cofactor of v and v' shows that all groups formed by means of these substitutions are transformable into the one group.

On the other hand, we have two distinct groups corresponding to $(A_1 A_2 A_3 A_4 A_5)$ all, the one consisting of both odd and even substitutions and generated by the aid of w and w', the other consisting entirely of even substitutions and generated by the aid of $a_1^5 a_2^5 . w$ and $a_1^2 a_2^2 . w'$.

From this head we have, therefore, the eight following groups:

$\{H_{243}, s\}$ of order 1215,
$\{H_{243}, s, t\}$ of order 2430_2,
$\{H_{243}, s, a_1^1 a_2^1 . a_1^2 a_2^2 . a_1^3 a_2^3 . a_1^4 a_2^4 . a_1^5 a_2^5 . t\}$ of order 2430_3,
$\{H_{243}, s, u\}$ of order 4860_2,
$\{H_{243}, s, a_1^1 a_2^1 . a_1^2 a_2^2 . a_1^3 a_2^3 . a_1^4 a_2^4 . a_1^5 a_2^5 . u\}$ of order 4860_3,
$\{H_{243}, v, v'\}$ of order 14580,
$\{H_{243}, w, w'\}$ of order 29160_2,
$\{H_{243}, a_1^5 a_2^5 . w, a_1^2 a_2^2 . w'\}$ of order 29160_3.

V. H_6 furnishes one group corresponding to each transitive group of degree 5. These groups are generated respectively by the substitutions s, t, u, v, v', w, w', and can readily be seen to be indentical with those of orders 30_2, 60_2, 120, 360_2, 720 included among the groups with three systems of imprimitivity. An interchange of suffixes and indices in the one set of groups gives the generating substitution of the other set of groups.

VI. H_3 furnishes groups corresponding to the transitive groups of degree 5 by means of the substitutions s, t, u, v, v', w, w'. As in the last case, however, these correspond to the groups of orders 15, 30_1, 60_1, 180, 360_1 included in the groups with three systems of imprimitivity. By the use of the cofactor $\sigma = a_1^1 a_2^1$.

$a_1^2a_2^2 . a_1^3a_2^3 . a_1^4a_2^4 . a_1^5a_2^5$ three more groups can be found generated respectively by the help of $t_1 = \sigma t$, $u_1 = \sigma u$, $w_1 = \sigma w$, $w_1' = \sigma w'$. These groups, however, are seen to be identical with those of orders 30_3, 60_3, and 360_3 included in the groups with three systems of imprimitivity. This head gives, therefore, no group essentially new.

VII. In the discussion of the head unity a useful theorem is the following given by Frobenius (Crelle t. CI, p. 287):

The average number of elements in all the substitutions of a group is $n - \alpha$, n *being the degree of the group, and* α *the number of its transitive constituents.*

The only transitive groups of degree 5 containing 15 as a factor of the order are the symmetric and alternating groups. We have therefore to find an imprimitive group of degree 15 with 5 systems of intransitivity simply isomorphic to the alternating (symmetric) group in 5 letters.

In determining the imprimitive group corresponding to $(A_1A_2A_3A_4A_5)$ pos, we make use of the following facts: (1) the 15 conjugate substitutions corresponding to terms of the type $A_1A_2 . A_3A_4$ must be of degrees 12 or 14; (2) the 20 conjugate substitutions corresponding to terms of the type $A_1 A_2 A_3$ must be of degrees 9, 12, or 15; (3) the 24 conjugate substitutions corresponding to terms of the type $A_1A_2A_3A_4A_5$ must be of degree 15. It must, therefore, be possible to solve the equation

$$15\,(12 + 2\alpha) + 20\,(9 + 3\beta) + 24.15 = 14.60$$

where $\alpha = 0, 1$; $\beta = 0, 1, 2$. The only solution is $\alpha = 0$, $\beta = 2$.

Therefore the imprimitive group we are seeking contains among its substitutions 15 of degree 12 and order 2, 20 of degree 15 and order 3, 24 of degree 15 and order 5. Making use of the relations among the generating substitutions of such a group of order 60 as given in Burnside, Theory of Groups, p. 107, we find that the two substitutions corresponding to $A_1A_2A_3A_4A_5$, $A_1A_2A_3A_4^3$, substitutions which will generate $(A_1A_2A_3A_4A_5)$ pos, are respectively,

$$s = a_1^1a_1^2a_1^3a_1^4a_1^5 . a_2^1a_2^2a_2^3a_2^4a_2^5 . a_3^1a_3^2a_3^3a_3^4a_3^5,$$
$$\rho = a_1^1a_3^2 . a_1^3a_2^4 . a_2^1a_1^2 . a_2^3a_3^4 . a_3^1a_2^2 . a_3^3a_1^4;$$

s and ρ are therefore the generating substitutions of an imprimitive group simply isomorphic to the alternating group of degree 5.

In determining a group simply isomorphic to $(A_1A_2A_3A_4A_5)$ all, we argue as before in regard to the various sets of conjugate substitutions. The 15 substitu-

tions corresponding to terms of the type $A_1A_2 \,.\, A_3A_4$ are of degrees 12 or 14, the 20 corresponding to the type $A_1A_2A_3$ are of degrees 9, 12, or 15, the 24 corresponding to the type $A_1A_2A_3A_4A_5$ are of degree 15, the 10 corresponding to the type A_1A_2 are of degrees 6, 8, 10, or 12; the 30 corresponding to the type A_1A_2 A_3A_4 are of degrees 12 or 14; the 20 corresponding to the type $A_1A_2A_3A_4A_5$ are of degree 15. The equation to be satisfied is therefore

$$15\,(12+2\alpha)+20\,(9+3\beta)+24.15+10\,(6+2\gamma)+30\,(12+2\delta)+20.15 = 14.120 \text{ where } \alpha=0,\ 1\,;\ \beta=0,\ 1,\ 2\,;\ \gamma=0,\ 1,\ 2,\ 3\,;\ \delta=0,\ 1.$$

The only solution is $\alpha=0$, $\beta=2$, $\gamma=3$, $\delta=1$. The substitutions $A_1A_2A_3A_4A_5$, $A_2A_3A_4A_5$, $A_1A_2A_3A_4$ will generate the group $(A_1A_2A_3A_4A_5)$ all, and corresponding to these as generators of the imprimitive group we have the three substitutions,

$$s = a_1^1a_1^2a_1^3a_1^4a_1^5 \,.\, a_2^1a_2^2a_2^3a_2^4a_2^5 \,.\, a_3^1a_3^2a_3^3a_3^4a_3^5,$$
$$\sigma = a_2^1a_3^1 \,.\, a_1^2a_1^3a_1^5a_1^4 \,.\, a_2^2a_2^3a_2^5a_3^4 \,.\, a_3^2a_2^3a_2^5a_3^4.$$
$$\rho = a_1^1a_3^2 \,.\, a_1^3a_2^4 \,.\, a_2^1a_1^2 \,.\, a_2^3a_3^4 \,.\, a_3^1a_2^2 \,.\, a_3^3a_1^4.$$

To sum up the results of the preceding work, the 16 heads with three systems of intransitivity give 41 groups with three systems of imprimitivity. The 7 heads with five systems of intransitivity give 42 groups with five systems of imprimitivity, but of these 13 groups contain also three systems of imprimitivity. Therefore there are 70 imprimitive groups of degree 15 as determined in this paper.

Primitive Substitution Groups of Degree Eighteen.

The main theorems employed in this investigation of primitive groups are the following, in which p is always to stand for a prime number.

I. *The order of a primitive group of degree n cannot exceed* $\dfrac{n\,!}{2.3\,\ldots\,p}$, *where* 2, 3, ... p *are the distinct primes which are less than* $\frac{2}{3}\,n$. (Burnside, Theory of Groups, p. 199).

II. *A group of degree $p+x$ or of degree $2p+x$, $x>2$, cannot be more than x times transitive.* (Miller, Bull. A. M. S., v. IV, pp. 142, 143).

III. *If a primitive group of degree n contains a circular substitution of prime*

order p, the group is at least $(n-p+1)$-fold transitive. (Cole's tr. of Netto's Theory of Substitutions, p. 93).

IV. *A self-conjugate subgroup of a primitive group must be transitive.* (Burnside, l. c., p. 187).

V. *A self-conjugate subgroup of a $\varkappa$-ply transitive group of degree n $(2 < \varkappa < n)$ is in general at least $(\varkappa - 1)$-ply transitive. The only exception is that a triply transitive group of degree 2^m may have a self-conjugate subgroup of order 2^m.* (Burnside, l. c., p. 189).

VI. *A group G which is at least doubly transitive either must be simple or mus contain a simple group H as a self-conjugate subgroup. In the latter case no operation of G except identity is permutable with every operation of H. The only exceptions to this statement are that a triply transitive group of degree 2^m may have a self conjugate subgroup of order 2^m, and that a doubly transitive group of deyree p^m may have a self-conjugate subgroup of order p^m.* (Burnside, l. c., p. 192).

VII. *The substitutions of a transitive group G which leave a given symbol unchanged form a maximal subgroup G_1, which is one of a set of n conjugate subgroups, each leaving one of the n elements unaffected.* (Burnside, l. c., p. 140).

VIII. *The number of substitutions of degree $l < n$ contained in a transitive group of degree n is equal to the number of substitutions of this same degree l contained in the maximal subgroup G_1 of degree $n-1$ multiplied by $\frac{n}{n-l}$.* (Stated by Miller, Quar. Jour. of Math. v. XXVIII, p. 215.)

IX. *The average number of elements in all the substitutions of a group is $n - \alpha$, n being the degree of the group and α the number of its transitive constituents.* (Frobenius, Crelle, t. CI, p. 287.)

X. Sylow's theorem, as stated by Burnside, l. c., p. 92, or by Sylow, "Théorèmes sur les groupes de substitutions," Math. Ann., v. V (1872), pp. 584 et seq.

XI. *The class of a primitive group of degree n is the same as the class of its maximal subgroup that leaves one element unaffected.*

While the preceding theorems are used throughout the work on primitive groups, the following are used mainly in the determination of simply transitive primitive groups.

XII. *A simply transitive primitive group G of degree n cannot contain a transitive subgroup of degree less than n.* (Miller, Quar. Jour. of Math., v. XXVIII, p. 215.

XIII. *When G_1 contains a self-conjugate subgroup H of degree $n-\alpha$, H must be intransitive, and it must be the transform with respect to substitutions of G of any one of $\alpha-1$ other subgroups of G_1 (H_1', H_2', ... $H_{\alpha-1}'$).* (Miller, Proc. Lon. Math. Soc., v. XXVIII, p. 534.)

XIV. *All the prime numbers which divide the order of one of the transitive constituents of G_1 divide also the orders of each of the other transitive constituents.*

Corollary I. *If one of the transitive constituents of G_1 is of a prime degree, each of its other transitive constituents is of the same or a larger degree, and the order of G_1 is the same as the order of the group formed by these other transitive constituents.*

Corollary II. *If the order of G_1 is not divisible by the square of a prime number, all its transitive constituents are of the same order, and G_1 is formed by establishing a simple isomorphism between them.* (Miller, l. c., p. 536.)

XV. *If a transitive constituent of G_1 is of a prime order, the order of G_1 is the same prime number, and G is of class $n-1$.*

Corollary. *If G_1 contains a constituent of degree 2, its order is 2, and the degree of G is a prime number.* (Miller, l. c., p. 536.)

The above theorems are given in the form and with the symbols most convenient for use, and so are not always exact quotations from the papers and books referred to, while the references given are not always references to the original paper in which the theorem appeared.

Applying these theorems now to the special case in which $n = 18$, we proceed as follows:

Since $18 = 2.7 + 4$, by Theorem II a primitive group cannot be more than 4-ply transitive.

By Theorem I the order is seen not to exceed

$$\frac{18!}{2.3.5.7.11} = 2^{15}.3^{7}.5^{2}.7.13.17.$$

If the group included circular substitutions of orders 2, 3, 5, 7, 11, 13, it would be at least 17, 16, 14, 12, 8, 6-fold transitive respectively according to Theorem III. This is impossible; therefore circular substitutions of these orders are not present, and consequently we see at once that 11 and 13 cannot be factors of the order.

If the order includes the factor 7, then, by Theorem X, there is a subgroup of order 7. This must consist of the powers of a substitution composed of two cycles of 7 elements each, and it must be contained self-conjugately in a group of order $7 . 4m$ that interchanges transitively among themselves the four remaining elements. (Cf. Burnside, Theory of Groups, p. 202.) It is quite possible to establish a $(7\alpha, 1)$ isomorphism between an imprimitive group of degree 14 with the systems of imprimitivity 7, 7 and a transitive group of degree 4; therefore 7 may be a factor of the order.

A subgroup of order 5^2 cannot be present, as it would have to be intransitive with the systems of intransitivity 5, 5 or 5, 5, 5. In the one case, it would have to be contained self-conjugately in a group of order $5^2.8m$, in the other, in a group of order $5^2.3.m$. In either case, a circular substitution of order 5 would be present, which is impossible.

The factor 5 may be contained in the order, as it is possible to establish a (5, 1) isomorphism between the cyclical group of degree 15 and the cyclical group in the remaining three letters.

The order must, therefore, be a factor of $2^{15}.3^{7}.5.7.17$.

Simply Transitive Groups.

The maximal subgroup G_1 that leaves a_1 unaffected is intransitive (Theorem XII), and its order is, therefore, a factor of $2^{14}.3^{5}.5.7$. Moreover, its class cannot be less than 6, for if it were 2, 3 or 5, G_1 would necessarily contain a transitive subgroup of too low a degree, and it cannot be of class 4, if G is to be primitive. (Netto, l. c., p. 138.)

By Theorem XIV, Cor. I, it is evident that G_1 cannot contain a transitive constituent of degree 13 or 11; by Theorem XIV it cannot contain a transitive

constituent of degree 15 or 14, and by Theorem XV, Cor., it cannot contain a constituent of degree 2.

If one of the transitive constituents of G_1 is of degree 12, the other must be of degree 5. The isomorphism between the transitive groups of degrees 12 and 5 must be an $(\alpha, 1)$ isomorphism, where α itself may be equal to 1. By Theorem XIV, the order of the group of degree 5 must contain the factors 5, 3, 2; therefore this group must be either the alternating or the symmetric group of degree 5. If the isomorphism is more than simple, then the group of degree 12 must be an imprimitive group with 6 systems of imprimitivity. The head for such an imprimitive group as we require is the intransitive group of order 2 and degree 12 given by $[1, a_1a_2 . a_3a_4 . a_5a_6 . a_7a_8 . a_9a_{10} . a_{11}a_{12}]$. The group $(a_{13}a_{14}a_{15}a_{16}a_{17})$ pos contains

24 conjugate substitutions of order 5 and degree 5,
20 conjugate substitutions of order 3 and degree 3,
15 conjugate substitutions of order 2 and degree 4.

Corresponding to these in the group of degree 17, we have 1 substitution of degree 12 and order 2, 24 of degree $15 + 2\alpha$ and order 5 or 10, 24 others of degree $15 + 2\alpha'$ and order 5 or 10, 20 of degree $15 + 2\beta$ and order 3 or 6, together with 20 of degree $15 + 2\beta'$ and order 3 or 6, 15 of degree $12 + 2\gamma$ and order 2 or 4, and 15 of degree $12 + 2\gamma'$ and order 2 or 4, where

$$\alpha = 0, 1; \ \alpha' = 0, 1; \ \beta = 0, 1; \ \beta' = 0, 1; \ \gamma = 0, 1, 2; \ \gamma' = 0, 1, 2.$$

By Theorem IX, the following equation must be satisfied:

$$12 + 24(15 + 2\alpha) + 24(15 + 2\alpha') + 20(15 + 2\beta) + 20(15 + 2\beta') + 15(12 + 2\gamma) + 15(12 + 2\gamma') = 120 . 15.$$

The only type of solution is given by $\alpha = \beta = \gamma = \beta' = 0$, $\alpha' = 1$, $\gamma' = 2$. G_1, therefore, contains both a self-conjugate subgroup of degree 12 and order 2, and 15 conjugate subgroups of the same type. But, by Theorem XIII, only 5 such conjugate subgroups should exist if this group is to be the G_1 of a simply transitive primitive group. This intransitive group gives us, therefore, no such group as we require.

For precisely the same reason the intransitive group formed by establishing a (2, 1) isomorphism between an imprimitive group of degree 12 and order 240,

and the symmetric group of degree 5 cannot be employed in the formation of a simply transitive primitive group of degree 18.

If the isomorphism is simple, the group G_1, including the alternating group of degree 5, must contain 24 substitutions of degree 10 or 15 and order 5, 20 of degree 6, 9, 12 or 15 and order 3, 15 of degree 6, 8, 10, 12, 14 or 16 and order 2. The following equation must, therefore, be satisfied: $24(10+\alpha)+20(6+\beta)+15(6+\gamma)=60\times 15$, where $\alpha=0, 5$; $\beta=0, 3, 6, 9$; $\gamma=0, 2, 4, 6, 8, 10$. The only solution is $\alpha=5$, $\beta=9$, $\gamma=10$. G_1 is, therefore, of class 15.

A group G of degree 18, formed with the help of this G_1 and, therefore, of order $60\cdot 18$, contains 36 conjugate subgroups of order 5, each of which is contained self-conjugately in a group of order 30. As each of these subgroups of order 5 is already self-conjugate in a group of order 10, the construction of the generating substitutions of such a group is an easy matter. G_1 is generated by

$$s = a_1a_3a_7a_5a_9\,.\,a_2a_4a_8a_6a_{10}\,.\,a_{13}a_{14}a_{15}a_{16}a_{17}$$

and by

$$t = a_1a_3\,.\,a_2a_4\,.\,a_5a_{12}\,.\,a_7a_8\,.\,a_9a_{10}\,.\,a_6a_{11}\,.\,a_{13}a_{15}\,.\,a_{14}a_{17},$$

and contains, as one of the above-mentioned groups of order 10, the group generated by

$$u = st = a_3a_8a_{11}a_6a_9\,.\,a_4a_7a_{12}a_5a_{10}\,.\,a_{13}a_{17}a_{15}a_{16}a_{14},$$

$$v = s^{-1}ts = a_1a_2\,.\,a_3a_7\,.\,a_4a_8\,.\,a_5a_6\,.\,a_9a_{12}\,.\,a_{10}a_{11}\,.\,a_{13}a_{15}\,.\,a_{14}a_{16}.$$

The group $\{u, v\}$ is a subgroup of a group of degree 18 and order 30 formed by establishing a (5, 1) isomorphism between an imprimitive group of degree 15 and order 30 with u and its powers as head and the symmetric group in the three elements $a_1a_2a_{18}$. The question then reduces to that of the determination of a substitution of degree 18 and order 3 that will transform the head $\{u\}$ into itself, interchange cyclically the three systems of $\{u\}$, and be in its turn transformed into its square by v. An examination of all substitutions fulfilling these conditions results in finding none that do not give, when combined with other substitutions of G_1, substitutions that cannot possibly belong to a simply transitive primitive group containing G_1 as a maximal subgroup.

There is no primitive or imprimitive group of degree 12 simply isomorphic to the group $(abcdef)_{120}$; consequently, no isomorphism can be established between the symmetric group of degree 5 and a transitive group of degree 12.

There remains the question whether the symmetric group of degree 5 can be put in a simply isomorphic relation to one of the imprimitive groups of degree

12 and order 120 that have both six and two systems of imprivitivity. Such groups of degree 12, however, contain two self-conjugate subgroups of orders 2 and 60 respectively, and, therefore, are not in a simply isomorphic relation to the symmetric group of degree 5.

If one of the transitive constituents is of degree 10, the other can only be of degree 7. By Theorem XIV, the group of degree 10 must contain 7 as a factor of its order, and, therefore, must be either the alternating or the symmetric group. It is impossible to establish an isomorphic relation between either of these groups and one of degree 7 without introducing substitutions of too low a degree.

If one of the transitive constituents is of degree 9, the remaining constituent may be either intransitive in two systems of four elements each or intransitive in eight elements. The isomorphism can in neither case be simple, as an examination of all groups of degree 8 and orders equal to those of transitive groups of degree 9 shows that in each case a system of intransitivity of degree 2 enters, with the single exception of a group of order 144. Here, however, the group of degree 9 contains a substitution of order 8, while an inspection of the corresponding groups of degree 8 shows no substitution of that order.

The isomorphism is, therefore, an (α, β) isomorphism, where α and β are not simultaneously equal to one.

When neither α nor β is equal to one, G_1 must be formed from an imprimitive group of degree 9 and an intransitive group of degree 8. The order of each transitive constituent must contain 3 as a factor, and, therefore, the group of degree 8 must be some combination of the alternating and symmetric groups of degree 4 in two systems of elements. The only combinations possible, consistent with the requirements of class, are got by establishing a simple isomorphism between the two symmetric groups of degree 4 or between the two alternating groups of the same degree. Every relation of isomorphism established between these groups of degree 8 and any imprimitive groups of degree 9 consistent with the requirements of class, results in a G_1 that contains a self-conjugate subgroup of order 4 and degree 8, and no other subgroups of the same order. This case, therefore, gives no simply transitive primitive group.

When α becomes 1, the group of degree 8 must, as before, be composed of either the symmetric or the alternating groups of degree 4 in two sets of elements put into the relation of simple isomorphism. The order of such a group does not contain 9 as a factor; therefore this case gives no possible G_1.

When β becomes 1, the group of degree 9 must be imprimitive. No transitive group of degree 8 stands, however, in the given relation of isomorphism towards an imprimitive group of degree 9. The only permissible intransitive groups of degree 8 are combinations of the symmetric and alternating groups of degree 4 in two sets of elements, and none of these are isomorphic in the given way to any imprimitive group of degree 9.

If one of the transitive constituents is of degree 8, we may have the systems 8, 6, 3 or 8, 3, 3, 3. In both cases we have an $(\alpha, 1)$ isomorphism between an intransitive group of degree 14 and the symmetric group of degree 3. The group of degree 8 is not primitive, as no suitable isomorphic relation can be established between a primitive group of degree 8 and an imprimitive or an intransitive group of degree 6. The only imprimitive groups of degree 8 that can be used are those with the head $(1, a_1a_2 . a_3a_4 . a_5a_6 . a_7a_8)$ that are isomorphic to a group of degree 4 and order 12 or 24. Such groups, however, cannot be combined with the groups in the remaining 9 elements in such a way as to generate a group capable of being the G_1 of one of the required primitive groups.

The case in which G_1 contains a transitive constituent of degree 7 has already been discussed, as according to Theorem XIV, Cor. I, the remaining constituents must be of larger degree.

If G_1 contains a transitive constituent of degree 6, the systems may be either 6, 6, 5 or 6, 4, 4, 3. For the former system the only possible arrangement is to establish a simple isomorphism between the three groups $(a_1a_2a_3a_4a_5a_6)_{60}$, $(a_7a_8a_9a_{10}a_{11}a_{12})_{60}$, $(a_{13}a_{14}a_{15}a_{16}a_{17})$ pos, or between the groups $(a_1a_2a_3a_4a_5a_6)_{120}$, $(a_7a_8a_9a_{10}a_{11}a_{12})_{120}$, $(a_{13}a_{14}a_{15}a_{16}a_{17})$ all. An examination of the two groups G_1 formed from these isomorphisms shows that these are not the maximal subgroups of simply transitive primitive groups of degree 18.

If the systems are 6, 4, 4, 3, only the imprimitive groups of degree 6, the alternating and symmetric groups of degree 4, and the symmetric groups of degree 3 are involved. The group formed by the system 6, 4, 4 has an $(\alpha, 1)$ isomorphism to the group of degree 3, and this isomorphism cannot be simple. No combination of these groups can be found fulfilling all the necessary conditions.

The case in which G_1 contains a transitive system of degree 5 has already been discussed, as the remaining systems must be of degree greater than 5.

If G_1 contains a transitive system of degree 4, the only arrangement possible is 4, 4, 3, 3, 3. The groups involved are therefore the symmetric groups of degree 3 and 4, and the alternating group of degree 4. One group consistent with the

requirements of class is got by establishing a (1, 4) isomorphism between the group,

$$(a_9a_{10}a_{11}\,.\,a_{12}a_{13}a_{14}\,.\,a_{15}a_{16}a_{17}) \text{ all, and } (a_1a_2a_3a_4\,.\,a_5a_6a_7a_8) \text{ all.}$$

This group contains, however, one and only one subgroup of degree 8 and order 2.

A second group is got by first establishing a (4, 1) isomorphism between $(a_1a_2a_3a_4.a_5a_6a_7a_8)$ all and $(a_9a_{10}a_{11})$ all; and then establishing a (12, 3) isomorphism between the group of order 24 so formed and the group $(a_{12}a_{13}a_{14}\,.\,a_{15}a_{16}a_{17})$ all. This group of degree 17 contains only one subgroup of order 4 and degree 8; therefore it cannot become a G_1.

G_1 cannot contain only systems of degree less than four, as in such a case a system of degree 2 would have to enter.

There is, therefore, no simply transitive primitive group of degree 18. This result when joined to all other determinations of similar groups shows that there is no simply transitive primitive group of degree $p+1$, p a prime number and ≤ 17.

Multiply transitive groups.

Among the transitive groups of degree 17 five contain a self-conjugate subgroup of order 17. These are of order 17, 2.17, 4.17, 8.17, 16.17 respectively, while all excepting the first are of class 16.

If a primitive group of degree 18 and order 18.17 existed, such a group would contain 18 conjugate subgroups of degree 17. It would therefore contain 17 substitutions of degree 18 and 18.16 of degree 17. By Sylow's theorem since $18.17 = 2.3^2.17$, such a group contains either 1 or 34 subgroups of order 3^2. A subgroup of this order must be intransitive, therefore cannot be self conjugate, and it is impossible to form 34 subgroups of order 9 from 17 substitutions of degree 18. No such group of degree 18 exists.

A primitive group of degree 18 and order $18.17.2 = 2^2.3^2.17$ would contain among its substitutions 153 of class 16 and order 2, 288 of class 17 and order 17, 170 of class 18. This group must contain either 1, 4, or 34 conjugate subgroups of order 3^2. As before, a subgroup of this order cannot be self-conjugate, as it is intransitive. If there were 4 conjugate subgroups, each would be self-conjugate in a group of order $3^2.17$ involving all 18 letters and necessarily transitive. Such a group is non-existent. If there were 34 conjugate subgroups they must be of degree 18, and there are not enough substitutions of class 18 to form all these subgroups.

A primitive group of degree 18 and order $18.17.4 = 2^3.3^2.17$ contains among its substitutions 476 of degree 18, 459 of degree 16, 288 of degree 17. According to Sylow's theorem it contains either 1, 3, 9, 17, 51, or 153 conjugate subgroups of order 2^3. Now the group leaving one element unchanged contains 17 conjugate subgroups of degree 16 and order 4; therefore the group of degree 18 contains 153 distinct conjugate subgroups of order 4; therefore it contains 153 conjugate subgroups of order 2^3. Each of these is contained self-conjugately in no larger group.

The number of systems of intransitivity in any one is got from the following equation, where x denotes the number of substitutions of degree 18 and α the number of systems:

$$18x + 16\,(7 - x) = 8\,(18 - \alpha), \text{ where } \alpha \neq 1,\ x < 8.$$

There are two sets of solutions, either $x = 0$, $\alpha = 4$, or $x = 4$, $\alpha = 3$.

The group of degree 17 is generated by,

$$s = a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}$$
$$t = a_2a_{14}a_{17}a_5 \cdot a_3a_{10}a_{16}a_9 \cdot a_4a_6a_{15}a_{13} \cdot a_7a_{11}a_{12}a_8,$$

where t and its powers form a self-conjugate subgroup of the group of order 2^3 and degree 18 that is now under discussion. It is impossible to so connect the systems and introduce the remaining elements that the first solution may give the group of order 2^3. Making use of the second solution we have only to combine with the group generated by t a substitution of degree 18 that connects the two remaining elements by a transposition, and unites the cycles of t in pairs. The 153 groups of order 2^3 give in this way $153.4 = 612$ distinct substitutions of degree 18, while there are only 476 in the group. This group of degree 18 does not exist.

If there is a primitive group of order $18 \cdot 17 \cdot 2^3 = 2^4 \cdot 3^2 \cdot 17$, it contains 288 substitutions of degree 17, 1071 of degree 16, 1088 of degree 18. The group of degree 17 which is generated by

$$s = a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}$$

and

$$t = a_2a_{10}a_{14}a_{16}a_{17}a_9a_5a_3 \cdot a_4a_{11}a_6a_{12}a_{15}a_8a_{13}a_7,$$

contains 17 conjugate subgroups of degree 16 and order 8; therefore in the group of degree 18 there are 153 such conjugate subgroups, and each of these is self-conjugate in a group of order 2^4 and degree 18. Denoting by α the number of systems of intransitivity of this group of order 2^4, and letting x denote the number of substitutions of degree 18 contained in the group, we have the equation $18x + 16\,(15 - x) = 16\,(18 - \alpha)$, where $\alpha \neq 1$. There are two solutions, $x = 8$, $\alpha = 2$; $x = 0$, $\alpha = 3$. The first solution would involve a larger number of substi-

tutions of degree 18 than are actually present in the group under consideration. The second solution shows that the group must contain 153 conjugate subgroups of order 2^4 and degree 18 consisting of substitutions of class 16 only, and involving three systems of intransitivity. A substitution must therefore be combined with t that transforms t into one of its powers. and has its head in the group generated by t; moreover, this substitution must have as one of its cycles the transposition $(a_1\, a_{18})$, and must have systems of intransitivity apart from this cycle consistent with the systems of t. Such a substitution is $\sigma = a_2a_{10} \cdot a_3\, a_{14} \cdot a_9a_{17} \cdot a_5a_{16} \cdot a_6a_{13} \cdot a_7a_{11} \cdot a_8a_{12} \cdot a_1a_{13}$. The required group is therefore $\{s, t, \sigma\}$.

It is not necessary to prove that these three substitutions give a group of the required order, as such a group would be necessarily doubly transitive, and it is known that there is a doubly transitive group of degree 18 and of the required order. By the mode of construction of the substitutions, it is evident that there is only the one type of group of this degree and order.

Any primitive group of degree 18 and order $18 \cdot 17 \cdot 16$ contains 2312 substitutions of degree 18, 288 of degree 17, 2295 of degree 16. The group of degree 17 contains 17 conjugate cyclical subgroups of degree 16 and order 16, therefore, the group of degree 18 contains 153 subgroups of order 16, each of which is self-conjugate in one of 153 conjugate subgroups of order 32. Giving α and x the usual meanings, we find that the group of order 32 involves the equation $18x + 16\,(31 - x) = 32\,(18 - \alpha)$, where $\alpha \neq 1$. The only solution is $\alpha = 2$, $x = 8$; therefore, the group of degree 12 and order 32 must be intransitive with two systems of intransitivity, and must contain 8 substitutions of degree 18, 23 of degree 16. We have to add, therefore, to the cyclical group of degree 16, 8 substitutions of degree 16 and 8 of degree 18, all of them containing as one cycle the transposition of the remaining two letters.

The group of degree 17 and order $17 \cdot 16$ has as generators

$$s = a_1a_2a_3a_4a_5a_6a_7a_8a_9a_1 \cdot a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}$$

and

$$u = a_2a_4a_{10}a_{11}a_{14}a_6a_{16}a_{12}a_{17}a_{15}a_9a_8a_5a_{13}a_3a_7.$$

The substitution $\tau = a_1a_{13} \cdot a_4a_7 \cdot a_3a_{10} \cdot a_5a_{11} \cdot a_6a_8 \cdot a_9a_{16} \cdot a_{11}a_{13} \cdot a_{12}a_{15}$ generates with s and u the required group of degree 18 and order $17 \cdot 16 \cdot 18$. A triply transitive group of such an order is known to exist (Burnside, l. c., p. 158); so no further proof that $\{s, u, \tau\}$ is a group is necessary. It is easy to see that the even substitutions of the group just found form the simple group of order $18 \cdot 17 \cdot 8$.

The three remaining transitive groups of degree 17 each contains 120 conjugate subgroups of order 17. They are of orders 15 . 16 . 17, 15 . 16 . 17 . 2, 15 . 16 . 17 . 4 respectively.

The group of degree 18 and order 15 . 16 . 17 . 18 would necessarily contain 816 conjugate subgroups of order 5. Each is self-conjugate in a group of order 90 connecting the remaining three elements transitively. This group is intransitive with two transitive constituents, one of degree 15 and order 90, the other of degree 3. The first, however, is non-existent, therefore, the group of degree 18 is non-existent.

The two remaining groups also, if they can generate primitive groups of degree 18, would generate groups that each contain 816 conjugate subgroups of order 5. In the one case, we should have to make use of an intransitive group containing as a transitive constituent a group of degree 15 and order 180, in the other, the transitive constituent would enter as a group of degree 15 and order 360. Both of these groups are non-existent; therefore, the three groups of degree 17, at present under discussion, furnish us with no new groups of degree 18.

As the case now stands, the conclusion arrived at may be summed up as follows:

There are no simply transitive primitive groups of degree 18, and in addition to the symmetric and alternating groups, there are only two multiply transitive groups of this degree, viz., the two given by

$$\left.\begin{array}{l}(a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}),\\ (a_2a_{10}a_{14}a_{16}a_{17}a_9a_5a_3\,.\,a_4a_{11}a_6a_{12}a_{15}a_8a_{13}a_7),\\ (a_2a_{10}\,.\,a_3a_{14}\,.\,a_9a_{17}\,.\,a_5a_{16}\,.\,a_6a_{13}\,.\,a_7a_{11}\,.\,a_8a_{12}\,.\,a_1a_{18}),\end{array}\right\}\text{of order 2448,}$$

$$\left.\begin{array}{l}(a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}),\\ (a_2a_4a_{10}a_{11}a_{14}a_6a_{16}a_{12}a_{17}a_{15}a_9a_8a_5a_{13}a_3a_7),\\ (a_1a_{18}\,.\,a_4a_7\,.\,a_3a_{10}\,.\,a_5a_{14}\,.\,a_6a_8\,.\,a_9a_{16}\,.\,a_{11}a_{13}\,.\,a_{12}a_{15}),\end{array}\right\}\text{of order 4896.}$$

The second of these is triply transitive, and contains the first, which is doubly transitive and simple, as a self-conjugate subgroup.

The works consulted in the preparation of this paper have included, in addition to the standard works on the subject by Jordan, Serret, Netto, and Burnside, the following papers:

Askwith, "On Possible Groups of Substitutions that can be formed with 3, 4, 5, 6, 7 Letters Respectively." Quar. Jour. Math., v. XXIV (1890), pp. 111–167.

"On Groups of Substitutions that can be formed with Eight Letters." Quar. Jour. Math., v. XXIV (1890), pp. 263–331.

"On Groups of Substitutions that can be formed with Nine Letters." Quar. Jour. Math., v. XXVI (1892), pp. 79–128.

Cayley, "On Substitution Groups for Two, Three, Four, Five, Six, Seven, and Eight Letters." Quar. Jour. Math., v. XXV (1891), pp. 71–88, 137–155.

Cole, "List of the Substitution Groups of Nine Letters." Quar. Jour. Math., v. XXVI (1892), pp. 372–388.

"The Transitive Substitution Groups of Nine Letters." Bull. New York Math. Soc., v. II (1893), pp. 250–258.

"List of the Transitive Substitution Groups of Ten and of Eleven Letters." Quar. Jour. Math., v. XXVII (1894), pp. 39–50.

"Note on the Substitution Groups of Six, Seven, and Eight Letters." Bull. New York Math. Soc., v. II (1893), pp. 184–190.

Miller, "Intransitive Substitution Groups of Ten Letters." Quar. Jour. Math., v. XXVII (1894), pp. 99–118.

"List of Transitive Substitution Groups of Degree Twelve." Quar. Jour. Math., v. XXVIII (1896), pp. 193–231.

"Note on the Transitive Substitution Groups of Degree Twelve." Bull. Amer. Math. Soc., v. I (1894–1895), pp. 255–258.

"On the Transitive Substitution Groups of Degrees Thirteen and Fourteen." Quar. Jour. Math., v. XXIX (1898), pp. 224–249.

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PHILADELPHIA, *January* 1901.

LIFE.

I was born in Elizabeth, New Jersey, December 30, 1869. In 1890, I entered Bryn Mawr College, selecting, as my major studies, Mathematics and Latin. In 1894, I received the degree of A. B. from this college. The first semester of the following year, 1894–1895, was spent at Bryn Mawr College as a graduate student in Mathematics and Physics, the second semester, in teaching at a preparatory school. During the year 1895–1896 I held the Fellowship in Mathematics in Bryn Mawr College, remaining there the following year, 1896–1897, as a graduate student. During the year 1897–1898 I held the Mary E. Garrett European Fellowship from Bryn Mawr College, and spent the entire year at the University of Göttingen, where I attended the lectures of Professors Klein and Hilbert. I then returned to Bryn Mawr College, where I was Fellow by Courtesy in Mathematics during the year 1898–1899. In the spring of 1899 I passed the examination at Bryn Mawr College for the degree of Doctor of Philosophy. My major subject was Mathematics, pursued under the direction of Professors Scott and Harkness, while my double minor was Physics, pursued under the direction of Professor Mackenzie.

My gratitude is due to all the Professors under whom I have studied, and especially to Professor Harkness, under whose direction this paper was prepared.

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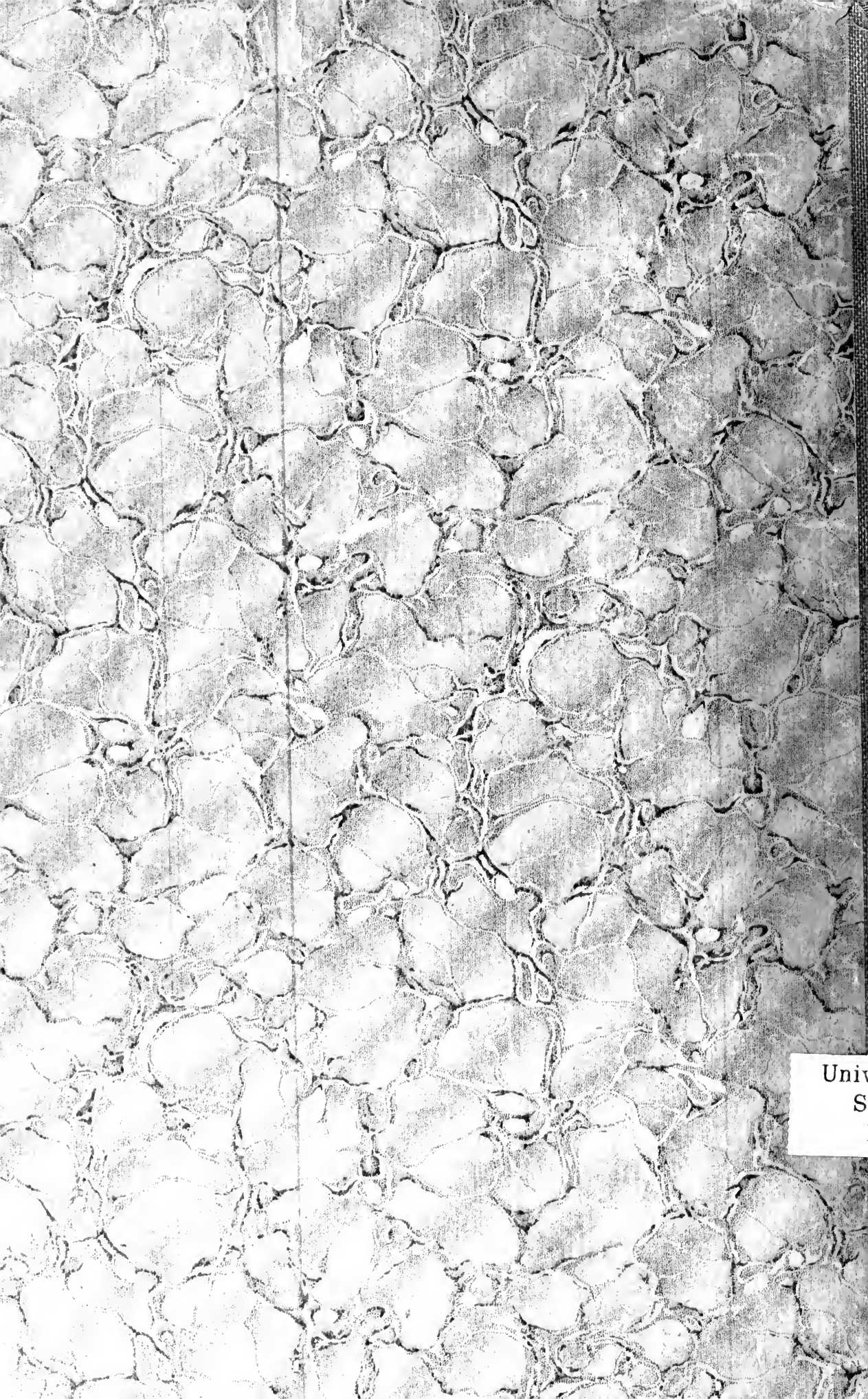

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